

John's position is not good for approximation

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Abstract

Recall that a convex body K is in John's position if the unit Euclidean ball is the maximal volume ellipsoid contained in K . Approximating convex body in John's position by polytopes we obtain the following results. 1. Let $n > R_n \geq 1$ be a sequence such that $\lim_{n \rightarrow \infty} \frac{R_n}{n} = 0$. For a sufficiently large n , we can construct a convex body $K \subset \mathbb{R}^n$ in John's position such that there is no P , polytope with a polynomial number of facets in n such that $K \subset P \subset R_n K$; 2. For a sufficiently large n , there is a convex body $K \subset \mathbb{R}^n$ in John's position such that there is no P , polytope that has less than $\exp(cn)$ facets satisfies $K \subset P \subset \sqrt{n}K$.

1 Introduction

One of the most natural questions in convex geometry is how well a convex body can be approximated by polytopes. The efficiency of approximation is measured by either the number of facets or the number of vertices of the polytope. There are different ways to measure how closely a polytope approximates a convex body. For instance, one can approximate a convex body K by an inscribed or a circumscribed polytope P and measure the difference between their volumes or mean-widths. In this paper, we are interested in one of the most natural measurements, the Banach-Mazur distance. For symmetric convex bodies, we have the following definition: Any K, L symmetric convex bodies in \mathbb{R}^n , the Banach-Mazur distance $d_{BM}(K, L)$ is defined by

$$d_{BM}(K, L) := \inf \{ r \geq 1, \exists T \in GL_n(\mathbb{R}) \text{ such that } K \subset TL \subset rK \}.$$

In particular, d_{BM} is a multiplicative distance for symmetric convex bodies. The problem of approximating by a polytope is well understood in the symmetric case. If $K = B_2^n$, which is the unit Euclidean ball, then a polytope

P with m facets will satisfy $d_{BM}(B_2^n, P) \geq c\sqrt{\frac{n}{\log(\frac{m}{n})}}$. This lower bound has obtained independently by 4 different groups of people adopted different approaches. (see [2],[4], [6] and [7])

Recently, A. Barvinok [3] has shown that for any symmetric convex body $K \subset \mathbb{R}^n$, one can find a polytope that has m facets such that $d_{BM}(K, P) = O(\sqrt{\frac{n \log(n)}{\log(m)}})$. In other words, the relation between the Banach-Mazur distance and the number of facets is well understood.

For the non-symmetric case, we have to modify the definition of the Banach-Mazur distance: For K, L convex bodies in \mathbb{R}^n , the Banach-Mazur distance $d_{BM}(K, L)$ is defined by

$$d_{BM}(K, L) := \inf \left\{ \begin{array}{l} r \geq 1, \exists T \in GL_n(\mathbb{R}) \text{ and } x, y \in \mathbb{R}^n \\ \text{such that } K - x \subset T(L - y) \subset r(K - x) \end{array} \right\}.$$

Unlike the symmetric case, the choice of the center is crucial. For a symmetric convex body, all natural centers such as the center of mass, the center of the John ellipsoid, the Santalo point etc., coincide with the center of symmetry. This is not the case for a general convex body. That is, given K and L , how does one choose the best center of scaling, x described in the definition of $d_{BM}(K, L)$? Moreover, it is not necessarily true that one of the classical invariant fixed points of the convex bodies is the best choice. For instance, for the non-symmetric MM^* problem, M. Rudelson [9] uses the center as an interlacing point between two classical invariant fixed points for the upper MM^* estimate.

There are two positive results in the non-symmetric case. Both of them uses the center of mass as the center of scaling. The best upper bound is provided by S. Szarek ([10]), which says that for any convex body in \mathbb{R}^n , there exists a polytope that has m facets or m vertices) such that $d_{BM}(K, P) \leq \frac{n}{\log(\frac{m}{n})}$. Using a random method, S. Brazitikos, G. Chasapis, and L. Hioni get an upper bound of order $\frac{n}{\sqrt{\log(\frac{m}{n})}}$ where m is the number of vertices. S. Szarek uses the center of mass of the approximating polytope P as the center of scaling. On the other hand, S. Brazitikos, G. Chasapis, and L. Hioni uses the center of mass of the convex body K as the center of scaling.

In this paper, we are examining the case when taking the center of scaling as the center of John ellipsoid of K . John ellipsoid of a convex body K is the unique maximized volume ellipsoid contained in K . A convex body is in John's position if its maximal volume ellipsoid is the unit Euclidean ball, B_2^n . For any convex body K , there exists an affine transformation so that it could map K to a convex body in John's position. If we consider convex bodies in

John's position, then, we automatically taking the center of its John ellipsoid as the origin.

The only known proof of showing that a symmetric convex body K can be approximated by a polytope P with polynomial in n facets so that $d(K, P) = O(\sqrt{n})$ relies on John's position. Using the contact points $\{x_i\}_{i=1}^m$ that form the identity decomposition (see definition below), we form the polytope $P := \{x \in \mathbb{R}^n, \langle x, x_i \rangle \leq 1 \ \forall i \leq m\}$ with $m = O(n^2)$. (Using the approximated John's decomposition, this can be improved to $m = O(n)$.) In particular, P satisfies,

$$B_2^n \subset K \subset P \subset \sqrt{n}B_2^n.$$

Thus, $d_{BM}(P, K) \leq \sqrt{n}$. Moreover, the result of A. Barvinok we mentioned before relies on John's decomposition as well. Although John's position exists for all convex bodies, the same construction in non-symmetric scenario can only provide

$$B_2^n \subset K \subset P \subset nB_2^n,$$

which shows that $d_{BM}(K, P) \leq n$. Given that so far the only means of obtaining a distance of $O(\sqrt{n})$ in the symmetric case is through John's position, one can ask whether John's position in the non-symmetric case can yield a better approximation. We have the following question:

Question 1.1. *Suppose a convex body $K \subset \mathbb{R}^n$ is in John's position. Is there a polytope P with a polynomial number of facets in n such that*

$$K \subset P \subset RP,$$

when $R = o(n)$?

In this paper, we answer this question negatively.

Theorem 1.2. *For a sufficiently large n and for any $c\sqrt{n} \leq R \leq c'n$, there exists a convex body $K \subset \mathbb{R}^n$ in John's position such that any polytope P satisfying*

$$K \subset P \subset RK,$$

has at least $\exp(C \log(\frac{R^2}{n}) \frac{n}{R^2})$ facets, where $c, c', C > 0$ are some universal constant.

Observe that the result of the main theorem is invariant under linear transformations but not under affine transformations. Thus, using the center of John ellipsoid of K as the scaling center is not good for approximation by polytopes with few facets.

An immediate consequence is the following:

Corollary 1.3. *Let $R_n \rightarrow +\infty$ be a positive increasing sequence that satisfies $\lim_{n \rightarrow \infty} \frac{R_n}{n} \rightarrow 0$. For any constant $C > 0$, there exists a convex body $K \subset \mathbb{R}^n$ in John's position for a sufficiently large n such that there is no polytope that has at most n^C number of facets that satisfy*

$$K \subset P \subset R_n K.$$

In the other extreme we have the following corollary:

Corollary 1.4. *For a sufficiently large n , there exists a convex body $K \subset \mathbb{R}^n$ in John's position such that there is no polytope P that has less than $\exp(cn)$ number of facets, and this satisfies*

$$K \subset P \subset \sqrt{n}K,$$

where $c > 0$ is a universal constant.

The fact that the theorem cannot provide a better result when $R = o(\sqrt{n})$ is not surprising. Using a net argument, one can derive the following:

Proposition 1.5. *Suppose $B_2^n \subset K \subset rB_2^n$. For a sufficiently small $\delta > 0$, there exists a polytope K_δ that has no more than $\exp(c \log(\frac{r}{\delta})n)$ facets such that*

$$(1 - 7\delta)K_\delta \subset K \subset K_\delta.$$

Applying the proposition to convex bodies in John's position we have

Corollary 1.6. *For a sufficiently large n , let K be a convex body in \mathbb{R}^n in John's position. Then, there exists a polytope P with $\exp(c \log(n)n)$ number of facets such that*

$$K \subset P \subset 2K,$$

where $c > 0$ is a universal constant.

Before proceeding to the proof, we examine the notation.

1.1 Preliminary

For the standard Euclidean space \mathbb{R}^n , let $\langle \cdot, \cdot \rangle$ denote the usual inner product. For a vector $x \in \mathbb{R}^n$, let $|x|$ denote its Euclidean norm. Let S^{n-1} be the unit sphere and B_2^n be the unit Euclidean ball. $GL_n(\mathbb{R})$ will be the collection of invertible linear transformations on \mathbb{R}^n .

A subset $K \subset \mathbb{R}^n$ is called a convex body if K is convex, compact, and has a non-empty interior. K is symmetric if it is symmetric with respect to

origin. ∂K will denote the boundary of K . For a convex body $K \subset \mathbb{R}^n$ that contains 0, we define its polar K° by

$$K^\circ := \{y \in \mathbb{R}^n, \forall x \in K \langle x, y \rangle \leq 1\}.$$

Indeed, duality will reverse the order of inclusion: If $K \subset L$, then $L^\circ \subset K^\circ$.

An ellipsoid $E \subset K$ is called the John ellipsoid if $\text{vol}(E) \geq \text{vol}(E')$ for any other ellipsoid $E' \subset K$. It is well known that the John ellipsoid of a convex body exists and is unique. Furthermore, for any convex body K , there exists an affine transformation T such that the John ellipsoid of TK is B_2^n .

Let $K \subset \mathbb{R}^n$ be a convex body in John's position. A point $x \in \mathbb{R}^n$ is a contact point of K and B_2^n if $x \in \partial K \cap \partial B_2^n$. A classical theorem of F. John provides a decomposition of identity in terms of contact points. ([1, p 52])

Theorem 1.7. *Let K be a convex body in \mathbb{R}^n that contains B_2^n . Then, K is in John's position if and only if there exist contact points x_1, \dots, x_m and $c_1, \dots, c_m > 0$ such that*

1. $\sum_{i=1}^m c_i x_i \otimes x_i = I_n$, and
2. $\sum_{i=1}^m c_i x_i = \vec{0}$.

Let Δ_n be the regular simplex in \mathbb{R}^n that has an inner radii equal to 1. Using the symmetry of Δ_n and uniqueness of the John ellipsoid (see [8]), it is not difficult to check that Δ_n is in John's position. Moreover, if u_1, \dots, u_{n+1} are contact points of Δ_n . Then, $\langle u_i, u_j \rangle = -\frac{1}{n}$ for $i \neq j$. Also, $-nu_i$ are the vertices of Δ_n .

For any integer $m \in \mathbb{N}$, we define $[m] := \{1, \dots, m\}$. For a subset $I \subset [m]$, let $|I|$ denote its cardinality. For $i, j \in \mathbb{N}$, we define

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

2 Proof

The construction of K uses both certain structures and randomness. K is obtained by intersecting a simplex with a large number of half spaces so that K is again a polytope. In this way, we start with a convex body in John's position. As long as each half space contains the John ellipsoid of Δ_n , K , too, will be in John's position.

The method described below shows how to determine if a convex body cannot be approximated by polytopes that have few facets with a fixed scaling point.

Proposition 2.1. Let $K := \{x \in \mathbb{R}^n, \langle x, y_i \rangle \leq 1 \ \forall i \in [m]\} \cap L$ where y_1, \dots, y_m are vectors in \mathbb{R}^n and L is a convex body in \mathbb{R}^n that contains 0. Suppose there are points $x_1, \dots, x_m \in K$ such that for some $R > 1$, we have

$$\langle x_i, y \rangle \begin{cases} = 1 & \text{if } y = y_i, \\ \leq \frac{1}{2R} & \text{if } y = y_j \text{ with } i \neq j. \\ \leq \frac{1}{2R} & \text{if } y \in L^\circ \end{cases}$$

Then, there is no polytope P that has less than $\frac{m}{2R}$ facets such that

$$K \subset P \subset RK.$$

Remark: Observe that, from the definitions of K , $y_i \in \partial K$ for $i \in [m]$.

Proof. Suppose there exists $w_1, \dots, w_{m_1} \in \mathbb{R}^n$ such that $P := \{x \in \mathbb{R}^n, \langle x, w_l \rangle \leq 1 \ \forall l \in [m_1]\}$ satisfies

$$K \subset P \subset RK.$$

The first inclusion indicates that all $w_l \in K^\circ$. The second inclusion is equivalent to the following: $\forall x \in \partial K, R\langle x, w_l \rangle \geq 1$ for some $l \in [m_1]$.

For $l \in [m_1]$, let O_l be the sub-collection of $\{x_i\}_{i=1}^m$ such that $R\langle x_i, w_l \rangle \geq 1$. Observe that $K^\circ = \text{conv}(\{y_i\}_{i=1}^m, L^\circ)$; thus, there exists $\lambda_i \geq 0$ for $i \in [m+1]$ and $y \in L^\circ$ such that w_l can be expressed in the following format:

$$w_l = \sum_{i=1}^m \lambda_i y_i + \lambda_{m+1} y.$$

This expression is not necessarily unique, but we fix one such expression. Then, according to the assumption on x_i ,

$$\begin{aligned} \langle w_l, Rx_i \rangle &= \sum_{j=1}^m \lambda_j \langle y_j, Rx_i \rangle + \lambda_{m+1} \langle y, Rx_i \rangle \\ &\leq \frac{1}{2} + \lambda_i R. \end{aligned}$$

Therefore, if $x_i \in O_l$, then $\lambda_i \geq \frac{1}{2R}$. Due to $\sum_{i=1}^m \lambda_i = 1$, we conclude that $|O_l| \leq 2R$.

Observe that $\cup_{l \in [m_1]} O_l = \{x_i\}_{i \in [m]}$; we conclude that $m_1 \geq \frac{m}{2R}$. Therefore, P has at least $\frac{m}{2R}$ facets. \square

The example K in the main theorem will be in the following form $K := \{x, \langle x, y_i \rangle \leq 1 \ \forall i \in [m]\} \cap \Delta_n$, where $\{y_i\}_{i=1}^m \subset \mathbb{R}^n$ and Δ_n is a regular simplex in John's position. Then, we will find $\{x_i\}_{i=1}^m$, which satisfies the assumption of Proposition 2.1.

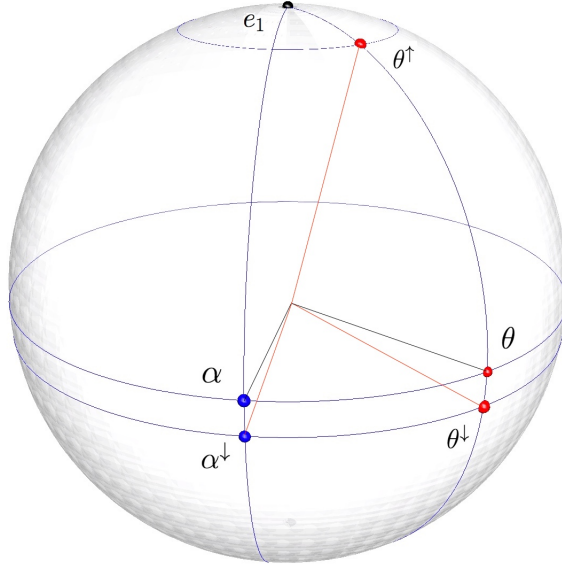
2.1 Structure

We need to pick y_i and x_i so that $\langle x_i, y_i \rangle = 1$. Yet, for $j \neq i$, $\langle x_i, y_j \rangle$ is small. Moreover, we also need to take into account of Δ_n . Thus, we need a certain structure here:

Proposition 2.2. *Let $S := S^{n-1} \cap \{x, \langle e_1, x \rangle = 0\}$. There exists some universal constant $t > 1, \frac{1}{t} > \epsilon > 0$ and $c_1 > 0$ that satisfy the following: For $\theta \in S$, we define $\theta^\downarrow := -\epsilon e_1 + \sqrt{1 - \epsilon^2} \theta$ and $\theta^\uparrow = \sqrt{1 - (\epsilon t)^2} e_1 + t \epsilon \theta$. Then, we have the following*

1. *For $\alpha, \theta \in S$, $\langle \alpha^\downarrow, \theta^\uparrow \rangle > 0$ implies $\langle \alpha, \theta \rangle > \frac{3}{4}$.*
2. *There exists a universal constant $c_1 > 0$ such that $\langle \theta^\uparrow, \theta^\downarrow \rangle = \frac{1}{c_1}$ for $\theta \in S$.*

Proof. The parameters ϵ, t, c_1 will be determined during the computations. We expect that ϵ is close to 0 and t is close to 1. The positions are demonstrated in the following figure:



We fix $\theta \in S$ and let $\alpha \in S$ satisfy $\langle \alpha, \theta \rangle = s \in [-1, 1]$. We can write $\alpha = s\theta + \sqrt{1 - s^2}\alpha'$ for some $\alpha' \in S$ and $\alpha' \perp \theta$. Using the orthogonality, we have

$$\begin{aligned} \langle \theta^\uparrow, \alpha^\downarrow \rangle &= \langle -\epsilon e_1 + \sqrt{1 - \epsilon^2}(s\theta + \sqrt{1 - s^2}\alpha'), \sqrt{1 - (\epsilon t)^2}e_1 + t\epsilon\theta \rangle \\ &= -\epsilon\sqrt{1 - (\epsilon t)^2} + \epsilon st\sqrt{1 - \epsilon^2}. \end{aligned}$$

As a consequence, the condition $\langle \theta^\uparrow, \alpha^\downarrow \rangle \geq 0$ is equivalent to $s \geq \frac{\sqrt{1-(\epsilon t)^2}}{t\sqrt{1-\epsilon^2}}$.

Furthermore, we set $t = \frac{8}{7}$. Then, $\frac{\sqrt{1-(\epsilon t)^2}}{t\sqrt{1-\epsilon^2}} \rightarrow \frac{7}{8}$ as $\epsilon \rightarrow 0$. Thus, for a sufficiently small $\epsilon > 0$, we have $\langle \theta^\uparrow, \alpha^\downarrow \rangle \geq 0 \Rightarrow \langle \alpha, \theta \rangle = s > \frac{3}{4}$.

To verify the second statement, we define $c_1 := \frac{1}{\langle \theta^\uparrow, \theta^\downarrow \rangle}$ for some $\theta \in S$. This definition is invariant under our choice of θ . It is sufficient to show that $\langle \theta^\uparrow, \theta^\downarrow \rangle > 0$.

$$\langle \theta^\downarrow, \theta^\uparrow \rangle = t\epsilon\sqrt{1-\epsilon^2} - \epsilon\sqrt{1-(\epsilon t)^2}.$$

Using $t > 1$, we have

$$\begin{aligned} & t\epsilon\sqrt{1-\epsilon^2} - \epsilon\sqrt{1-(\epsilon t)^2} \\ & > t\epsilon\sqrt{1-\epsilon^2} - \epsilon\sqrt{1-\epsilon^2} \\ & > 0. \end{aligned}$$

□

Proposition 2.2 will be used in the following way. After we specify what e_1 is, x_i will be θ^\downarrow for some $\theta \in S$ and y_i will be $c_1\theta^\uparrow$. In particular, the first statement of Proposition 2.2 will be used to guarantee that $\langle x_i, y_j \rangle$ is small if we choose x_i carefully.

2.2 Randomness

The goal here is to use randomness to pick $\theta \in S$ so that we can generate x_i and y_i . However, given Δ_n , we want to pick θ randomly yet guarantee that $\rho_{\Delta_n}(\theta^\uparrow)$ is large. Thus, the uniform randomness on S does not work in this case. The following lemma is a tail bound for hypergeometric distribution.

Lemma 2.3. *For a sufficiently large $n \in \mathbb{N}_+$, let k be some positive integer satisfying $1 < k < \frac{1}{2e^8}n$. Let \mathbb{P} be the uniform probability on the collection $\{I \subset [n], |I| = k\}$. Let I, J be 2 independent random sets that have the distribution \mathbb{P} . The probability that $|I \cap J| \geq \frac{k}{2}$ is less than $(\frac{2k}{n})^{k/5}$.*

Proof. We may assume that J is fixed. For any positive integer $1 \leq l \leq k$,

$$\mathbb{P}(|I \cap J| = l) = \frac{\binom{k}{l} \binom{n-k}{k-l}}{\binom{n}{k}}. \quad (2.2.1)$$

For positive integers $a \geq b$, $\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b(b-1)\cdots 1}$. A standard upper and lower estimate of $\binom{a}{b}$ is the following:

$$\left(\frac{a}{b}\right)^b \leq \binom{a}{b} \leq \left(\frac{ea}{b}\right)^b.$$

Applying these bounds to (2.2.1), we have

$$\begin{aligned} \mathbb{P}(|I \cap J| = l) &\leq \left(\frac{ek}{l}\right)^l \left(\frac{e(n-k)}{k-l}\right)^{k-l} \left(\frac{k}{n}\right)^k \\ &= e^k \left(\frac{k^2}{ln}\right)^l \left(\frac{n-k}{n}\right)^{k-l} \left(\frac{k}{k-l}\right)^{k-l}. \end{aligned} \tag{2.2.2}$$

Assuming that $l \geq \frac{k}{2}$ and $2k < n$,

$$\left(\frac{k^2}{ln}\right)^l \leq \left(\frac{2k}{n}\right)^l \leq \left(\frac{2k}{n}\right)^{\frac{k}{2}}.$$

Also, using $(1+x) \leq e^x$ for $x \in \mathbb{R}$ we have

$$\left(\frac{k}{k-l}\right)^{k-l} = \left(1 + \frac{l}{k-l}\right)^{k-l} \leq e^l \leq e^k.$$

Together with $\left(\frac{n-k}{n}\right)^{k-l} \leq 1$, we have

$$\begin{aligned} (2.2.2) &\leq \exp\left(k - \log\left(\frac{n}{2k}\right)\frac{k}{2} + k\right) \\ &= \exp\left(2k - \log\left(\frac{n}{2k}\right)\frac{k}{2}\right). \end{aligned}$$

If $\frac{n}{2k} \geq e^8$, then $2k \leq \log\left(\frac{n}{2k}\right)\frac{k}{4}$. We have the following:

$$\mathbb{P}(|I \cap J| = l) \leq \exp\left(-\log\left(\frac{n}{2k}\right)\frac{k}{4}\right).$$

Using a union bound, we may conclude the following statement: If $2e^8k \leq n$ and n is sufficiently large, then,

$$\mathbb{P}(|I \cap J| \geq \frac{k}{2}) \leq k \exp\left(-\log\left(\frac{n}{2k}\right)\frac{k}{4}\right) \leq \exp\left(-\log\left(\frac{n}{2k}\right)\frac{k}{5}\right) = \left(\frac{2k}{n}\right)^{k/5}.$$

□

2.3 Construction

Let Δ_n be the n -dimensional simplex in John's position and u_1, \dots, u_{n+1} be its contact points. We define $S := S^{n-1} \cap \{x \in \mathbb{R}^n, \langle x, u_1 \rangle = 0\}$. $\Delta'_n = \Delta_n \cap \{\langle x, u_1 \rangle = 0\}$ is a $n-1$ dimensional regular simplex and let $v_1, \dots, v_n \in S$, such that $c_n n v_i$ are vertices of Δ'_n , where $c_n \rightarrow 1$ as $n \rightarrow \infty$. Let $1 \leq k \leq n$. For $I \subset [n]$ with $|I| = k$, we define

$$u_I = \frac{\sum_{i \in I} v_i}{|\sum_{i \in I} v_i|}.$$

With this notation, we have the following proposition:

Proposition 2.4. *For a sufficiently large n , let $1 \leq k \leq \frac{n+3}{4}$. For $I, J \subset [n]$ with $|I| = |J| = k$ and $|I \cap J| < \frac{k}{2}$, we have*

$$\langle u_I, u_J \rangle \leq \frac{3}{4}.$$

Proof. With $\langle v_i, v_j \rangle = -\frac{1}{n-1}$ if $i \neq j$, we have

$$\begin{aligned} \langle \sum_{i \in I} v_i, \sum_{i \in I} v_i \rangle &= \sum_{i, j \in I} \left(-\frac{1}{n-1} + \delta_{ij} \left(1 + \frac{1}{n-1} \right) \right) \\ &= -\frac{k^2}{n-1} + k \left(1 + \frac{1}{n-1} \right) \\ &= k \left(1 - \frac{k-1}{n-1} \right). \end{aligned}$$

Thus,

$$u_I = \frac{c_{n,k}}{\sqrt{k}} \sum_{i \in I} v_i,$$

where $c_{n,k} = \frac{1}{\sqrt{1 - \frac{k-1}{n-1}}}$. In particular, if $1 \leq k \leq \frac{n+3}{4}$, then

$$1 \leq c_{n,k} \leq \sqrt{\frac{4}{3}}.$$

Let $J \subset [n]$ with $|J| = k$. Then,

$$\begin{aligned}
\frac{k}{c_{n,k}^2} \langle u_I, u_J \rangle &= \sum_{i \in I} \sum_{j \in J} \langle v_i, v_j \rangle \\
&= \sum_{i \in I} \sum_{j \in J} \left(-\frac{1}{n-1} + (1 + \frac{1}{n-1}) \delta_{ij} \right) \\
&= |I \cap J| \left(1 + \frac{1}{n-1} \right) - \frac{k^2}{n-1} \\
&\leq |I \cap J| \left(1 + \frac{1}{n-1} \right).
\end{aligned}$$

If $|I \cap J| < \frac{k}{2}$ and n is large enough, we get

$$\langle u_I, u_J \rangle \leq \frac{c_{n,k}^2}{2} \left(1 + \frac{1}{n-1} \right) \leq \frac{2}{3} \left(1 + \frac{1}{n-1} \right) < \frac{3}{4}.$$

□

Now we are ready to prove the main theorem.

Proof. Fix $1 \leq k \leq \frac{n}{2e^8}$. In particular, when n sufficiently large, our k satisfies the assumption Lemma 2.3 and Proposition 2.4. Let I_1 be a random set that is uniformly distributed on $\{I \subset [n], |I| = k\}$. Let $m \in \mathbb{N}$ be an integer that we will specify later. Let I_2, \dots, I_m be independent copies of I_1 . We choose the ϵ, δ, c_1 so that Proposition 2.2 holds, and let $e_1 = u_1$. Thus, S is defined as in Proposition 2.4. We adapt the definition of θ^\uparrow and θ^\downarrow for $\theta \in S$. Now, let

$$K = \Delta_n \cap \left(\bigcap_{i=1}^m \{x \in \mathbb{R}^n, \langle x, u_{I_i}^\uparrow \rangle \leq 1\} \right).$$

By Lemma 2.3, we have $\mathbb{P}(|I_i \cap I_j| \geq \frac{k}{2}) \leq (\frac{2k}{n})^{\frac{k}{5}}$. A union bound argument shows that

$$\mathbb{P}(\exists 1 \leq i < j \leq m \text{ such that } |I_i \cap I_j| \geq \frac{k}{2}) \leq \binom{m}{2} \left(\frac{2k}{n} \right)^{\frac{k}{5}} < m^2 \left(\frac{2k}{n} \right)^{\frac{k}{5}}.$$

By setting $m = (\frac{n}{2k})^{k/20}$ we have

$$\mathbb{P}(\exists 1 \leq i < j \leq m \text{ such that } |I_i \cap I_j| \geq \frac{k}{2}) \leq \left(\frac{2k}{n} \right)^{\frac{k}{10}}. \quad (2.3.1)$$

Since the probability is small than 1, there exists a sample such that $|I_i \cap I_j| < \frac{k}{2}$ for all $1 \leq i < j \leq m$. From now on, we fixed such sample.

We want to apply Proposition 2.1 with $L = \Delta_n$, $y_i = u_{I_i}^\uparrow$ and $x_i = c_1 u_{I_i}^\downarrow$. We start verifying the assumption that is described in Proposition 2.1.

First, $\Delta_n^\circ = \text{conv}\{u_1, \dots, u_{n+1}\}$. Recall that $\{c_n n v_i\}_{i=1}^m \subset \Delta_n$; we have $\langle c_n n v_i, u_j \rangle \leq 1$ for $i \in [n]$ and $j \in [n+1]$. Thus, for any $I \subset [n]$ with $|I| = k$ and $j \in [n+1]$ we have,

$$\langle u_I, u_j \rangle = \frac{c_{n,k}}{\sqrt{k}} \sum_{i \in I} \langle v_i, u_j \rangle \leq \frac{c_{n,k}}{\sqrt{k}} k \frac{1}{c_n n} \leq \frac{c_{n,k} \sqrt{k}}{c_n n}.$$

With $\langle -u_i, u_j \rangle = -(1 + \frac{1}{n})\delta_{ij} + \frac{1}{n} \leq \frac{1}{n}$,

$$\begin{aligned} \langle c_1 u_I^\downarrow, u_j \rangle &\leq c_1 \epsilon \langle -u_1, u_j \rangle + c_1 \sqrt{1 - \epsilon^2} \langle u_I, u_j \rangle \\ &\leq c_1 \epsilon \frac{1}{n} + \sqrt{1 - \epsilon^2} c_1 \frac{c_{n,k} \sqrt{k}}{c_n n}. \end{aligned}$$

With $\frac{n}{2e^8} \geq k \geq 1$, $1 \leq c_{n,k} \leq \sqrt{\frac{4}{3}}$ and $c_n \rightarrow 1$ as $n \rightarrow +\infty$, the inequality can be simplified to

$$\langle c_1 u_I^\downarrow, u_j \rangle \leq 3c_1 \frac{\sqrt{k}}{n}.$$

There is an expression $y = \sum_{i=1}^{n+1} \lambda_i u_i$ where $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$ for any $y \in \Delta_n^\circ$. We conclude that when n is large enough, $\forall y \in \Delta_n^\circ$ and $i \in [m+1]$,

$$\langle c_1 u_{I_i}^\downarrow, y \rangle \leq 3c_1 \frac{\sqrt{k}}{n}. \quad (2.3.2)$$

For $i, j \in [m+1]$ with $i \neq j$, due to Proposition 2.4, $|I_i \cap I_j| < \frac{k}{2}$ implies $\langle u_{I_i}, u_{I_j} \rangle < \frac{3}{4}$. According to Proposition 2.2, the last inequality indicates $\langle c_1 u_{I_i}^\downarrow, u_{I_j}^\uparrow \rangle < 0$. Moreover, by definition $\langle c_1 u_{I_i}^\downarrow, u_{I_i}^\uparrow \rangle = 1$. To summarize,

$$\langle c_1 u_{I_i}^\downarrow, y \rangle \begin{cases} = 1 & \text{if } y = u_{I_i}^\uparrow, \\ \leq 0 & \text{if } y = u_{I_j}^\uparrow \text{ with } j \neq i, \\ \leq 3c_1 \frac{\sqrt{k}}{n} & \text{if } y \in \Delta_n^\circ. \end{cases} \quad (2.3.3)$$

Now, we can apply Proposition 2.1 with $m = (\frac{n}{2k})^{k/10}$, $y_i = u_{I_i}^\uparrow$, $x_i = c_1 u_{I_i}^\downarrow$, $L = \Delta_n$ and $R = \frac{n}{6c_1 \sqrt{k}}$ with the condition that $1 \leq k \leq \frac{n}{2e^8}$. Rewriting everything in terms of R and n , we have

$$k = \left(\frac{n}{6c_1 R}\right)^2 \quad m = \left(\frac{18c_1^2 R^2}{n}\right)^{(\frac{n}{6c_1 R})^2/20} \quad \text{and} \quad \frac{\sqrt{2}e^4}{6c_1} \sqrt{n} \leq R \leq \frac{n}{6c_1}.$$

The lower bound on the facets of the polytope P in Proposition 2.1 is $\frac{m}{2R}$. To simplify m , we further restrict $R > c\sqrt{n}$ for some large $c > 0$, so that $\frac{R^2}{n} > 1$ and $\log(18c_1^2) > -\frac{1}{2}\log(\frac{R^2}{n})$. Thus,

$$\log\left(\frac{18c_1^2 R^2}{n}\right) = \log(18c_1^2) + \log\left(\frac{R^2}{n}\right) \geq \frac{1}{2}\log\left(\frac{R^2}{n}\right).$$

Then,

$$\begin{aligned} \frac{m}{2R} &= \exp(-\log(2R) + \log\left(\frac{18c_1^2 R^2}{n}\right) \frac{n^2}{720c_1^2 R^2}) \\ &\geq \exp(-\log(2R) + C' \log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2}) \\ &\geq \exp(-\log(2n) + C' \log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2}), \end{aligned}$$

for some $C' > 0$.

In order to take care the $\log(2n)$ term we need to check the later term carefully. First,

$$\frac{d}{dR} \log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2} = -\frac{2n^2}{R^3} \left(\log\left(\frac{R^2}{n}\right) - 1 \right) < 0$$

for $R > c\sqrt{n}$. Let $R = c'n$ and we will adjust c' later. We have

$$C' \log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2} = \frac{C'}{c'^2} \log(n) + \frac{C'}{c'^2} \log(c'^2).$$

Now, setting $1 > c' > 0$ so that $\frac{C'}{c'^2} > 8$ and $c' \leq \frac{1}{6c_1}$, for a sufficiently large n , we have

$$\frac{C'}{c'^2} \log(n) + \frac{C'}{c'^2} \log(c'^2) > 4 \log(n)$$

and

$$2 \log(n) \geq \log(2n).$$

Thus, $\frac{1}{2}C' \log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2} \geq \log(2n)$ when $R = c'n$. Since $\log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2}$ is a decreasing function for $R > \sqrt{n}$, $\frac{1}{2}C' \log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2} \geq \log(2n)$ for $c\sqrt{n} < R < c'n$. Therefore, we conclude that, for $c\sqrt{n} < R < c'n$,

$$\frac{m}{2R} \geq \exp\left(C \log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2}\right)$$

where $c, c', C > 0$ are some universal constants. Therefore, when n is sufficiently large, there exists a convex body $K \subset \mathbb{R}^n$ in John's position such that no polytope P that has facets less than $\exp\left(C \log\left(\frac{R^2}{n}\right) \frac{n^2}{R^2}\right)$ satisfies

$$K \subset P \subset RK.$$

□

3 Upperbound when R is small

Proposition 3.1. *Suppose $B_2^n \subset K \subset RB_2^n$. For a sufficiently small $\delta > 0$, there exists a polytope K_δ with no more than $\exp(c \log(\frac{R}{\delta})n)$ facets such that $(1 - 7\delta)K_\delta \subset K \subset K_\delta$.*

Proof. Let $B_2^n \subset K \subset RB_2^n$ be a convex body and $h : S^{n-1} \rightarrow [0, +\infty)$ be its support function. In particular, h is also the norm of K° , thus; h is a 1-Lipschitz continuous function and $1 \geq h(\theta) \geq \frac{1}{R}$ for $\theta \in S^{n-1}$.

Let \mathcal{N} be a ϵ -net of S^{n-1} for some sufficiently small $\epsilon \leq \frac{\delta}{R}$ where $1 > \delta_0 > \delta > 0$. We choose δ_0 small enough so that $\forall \theta \in S^{n-1}$, and there exists $\alpha \in \mathcal{N}$ such that $\langle \alpha, \theta \rangle \geq 1 - \epsilon^2$. (This does not depend on n , but it requires $\frac{\delta}{R}$ to be sufficiently small) Then, together with h is 1-Lipschitz, we have

$$\begin{aligned} \|h(\alpha)\alpha - h(\theta)\theta\|_2^2 &= h^2(\alpha) - 2h(\alpha)h(\theta)\langle \alpha, \theta \rangle + h^2(\theta) \\ &\leq h^2(\alpha) + (h(\alpha) + \epsilon)^2 - 2h(\alpha)h(\theta)\langle \alpha, \theta \rangle. \end{aligned} \quad (3.0.1)$$

With $h(\theta) \geq h(\alpha) - \epsilon \geq \frac{1}{R} - \epsilon > 0$, we have

$$2h(\alpha)h(\theta)\langle \alpha, \theta \rangle \geq 2h(\alpha)(h(\alpha) - \epsilon)(1 - \epsilon^2).$$

Thus, the previous inequality is bounded by

$$\begin{aligned} (3.0.1) &\leq 2h^2(\alpha) + 2\epsilon h(\alpha) + \epsilon^2 - 2h(\alpha)(h(\alpha) - \epsilon)(1 - \epsilon^2) \\ &= 2h^2(\alpha) + 2\epsilon h(\alpha) + \epsilon^2 - 2h^2(\alpha)(1 - \epsilon^2) + 2h(\alpha)\epsilon(1 - \epsilon^2) \\ &= 4\epsilon h(\alpha) + \epsilon^2(2h^2(\alpha) + 1) - \epsilon^3 2h(\alpha). \end{aligned}$$

With $h(\alpha) \leq 1$, we conclude that

$$(3.0.1) \leq 7\epsilon.$$

Therefore, for any $x \in \partial K$, there exists $y \in \partial K^\circ$, such that $\langle y, x \rangle = 1$. Since for any $y \in \partial K$, it is of the form $h(\theta)\theta$ for some $\theta \in S^{n-1}$, there exists $y' = h(\alpha)\alpha$ with $\alpha \in \mathcal{N}$ such that $\langle x, y' \rangle \geq 1 - \|x\|_2 \|y - y'\|_2 \geq 1 - R \frac{\delta}{R} \geq 1 - 7\delta$.

Recall that the size of a ϵ -net on S^{n-1} is bounded by $\exp(c \log(\frac{1}{\epsilon})n)$ for some $c > 0$. Therefore, if we define

$$K_\delta = \{x \in \mathbb{R}^n, \forall \theta \in \mathcal{N}, \langle h(\theta)\theta, x \rangle \leq 1\}.$$

Then we have $(1 - 7\delta)K_\delta \subset K \subset K_\delta$. □

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